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A Study of the NBS Time Scale Algorithm

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Abstract—**Since 1968 the NBS time scale algorithm has been generating a clock which is theoretically better than any of the individual clocks in its ensemble. In the last few years, thanks to the Global Positioning System, we have been able to measure the time difference between the NBS time scale algorithm and the other time standards around the world. We are able to study long-term stability of the order of years, and short-term stability of the order of days. We now have estimated fractional frequency stabilities for averaging times out to a year of about 1.5 $\times$ 10^{-14}.**

In this paper we study various aspects of the algorithm theoretically, comparing the NBS algorithm with a Kalman filter to discuss questions of optimality. We see that since we do not measure the time of a clock, but only the time difference between clocks, a time scale should not attempt to optimize time accuracy, since that has no meaning. However, time uniformity and frequency stability can be optimized.

We further study the practice of monitoring the clocks in a time scale for frequency steps, and removing a clock from the scale when a step has been detected until the new frequency is learned. We show that the effect of this practice on the algorithm is to translate random walk behavior in the individual clocks, due to the frequency steps of the clocks, to flicker noise for the ensemble. The implication here is that careful monitoring of the scale can significantly improve its long-term performance.

I. Introduction

A TIME SCALE algorithm can enable a time laboratory to increase the stability, accuracy, and reliability beyond the performance level of the physical clocks in its ensemble. The NBS time scale algorithm is a three-tiered process, estimating time, weight, and frequency for each clock at each measurement cycle [1]. It is an adaptive filter, adjusting weights in each measurement cycle according to the size of the time residuals. The equations of the algorithm are shown in the Appendix. In this paper we study various aspects of the algorithm, stating the assumptions for which the estimate are optimal, discussing the validity of those assumptions, and deriving aspects of the algorithm not heretofore published.

One interesting part of this study relates to an observation which has been made with real clocks concerning the long-term stability of the algorithm. In practice the clocks in the ensemble generating the time scale are carefully monitored. When a frequency step is detected in a clock, that clock is kept from contributing to the scale until the scale learns its new frequency. The effect of this practice on the algorithm is to translate random walk behavior in the individual clocks, from the frequency steps of the clocks, to flicker noise for the ensemble. This tendency has important implications for improving the long-term stability of an ensemble's net performance. This effect is investigated using simulation.

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II. Theoretical Analysis

We compare the NBS algorithm to the Kalman filter formalism. Time scales using a Kalman filter design have been studied elsewhere [2]–[6]. The purpose here is to use the theoretical least squares nature of the Kalman filter to compare to the NBS scale. First we study the structure of the NBS algorithm to verify its optimality for estimating the behavior of clocks. Second we consider the question of optimal weights. We find that, though the forms of the equations for predicting and updating clock estimates are identical, there is some disagreement over the weights the two methods consider optimal.

Both algorithms allow states of time and frequency, x and y, for the clocks, as well as considering a constant frequency drift, D, which is not estimated within either filter. The prediction forward in time is identical in the two algorithms. For each clock

$$x(n+1) = \hat{x}(n) + y(n) * \tau$$

$$y(n+1) = \hat{y}(n) + D * \tau.$$

The updates can be shown to be identical, if one matches the Kalman gains with weights and filter time constants in the NBS scale.

For the Kalman, if X is the n-clock state vector:

$$X = [x_1, y_1, \cdots, x_n, y_n]'$$

and K is the Kalman gain vector:

$$K = [k_1, k_2, \cdots, k_{2n-1}, k_{2n}]'$$

we may update sequentially $n - 1$ times, for each independent measurement x_{i1} = clock$_i$ − clock$_1$, using the measurement matrix h_i, defined by

$$H_i * \hat{X} = \hat{x}_i - \hat{x}_1.$$

Then the ith update is

$$X = \hat{X} + K_i * [x_{i1} - \hat{x}_i + \hat{x}_1]$$

where the ˆ denotes the previous best estimate, either the last prediction, if this is the first update, or the last update. Previous work has documented some stochastic models for the commercial cesium beam clocks in the NBS ensemble [2]. From this work it has been shown that a two-parameter model is efficient, namely, specifying the white noise frequency modulation (FM) level and the random walk FM level. We denote these as σ_ϵ and σ_η. This overall design for a Kalman filter estimating time from an ensemble of clocks has been used elsewhere [3]-[6].

For the NBS algorithm we update our estimate of the time of clock i against the scale, x_i, using the weights for each clock, w_i, and the measurements of clock i minus clock j: x_{ij}:

$$x_i = \sum_{j=1}^{n} w_j * (\hat{x}_j - x_{ij}).$$

We update our estimate of the frequency of clock i against the scale, y_i, using an exponential filter:

$$\hat{y}_i = \big(x_i(t + \tau) - x_i(t)\big)/\tau$$

$$y_i = (\hat{y}_i + N_i * y_i)/(1 + N_i).$$

In a two-clock system, the update equations are equivalent if we identify

$$w_1 = k_1 \qquad w_2 = -k_2$$

$$1/(N_1 + 1) = \tau * k_2/k_1$$

$$1/(N_2 + 1) = \tau * k_4/k_3.$$

In a three-clock system, from algebraic manipulations we find that the update equations are equivalent under an identification. Recall that we now have two updates, hence, two Kalman gain vectors. Let us denote

$$K_1 = [\hat{k}_1, \hat{k}_2, \cdots, \hat{k}_5, \hat{k}_6]'$$

and

$$K_2 = [k_1, k_2, \cdots, k_5, k_6]'.$$

Then there is a system of identifications for the time update as follows:

$$w_2 = \hat{k}_1 + k_1 * (\hat{k}_5 - \hat{k}_1), \qquad w_3 = k_1$$

$$w_1 + w_3 = -\hat{k}_3 - k_3 * (\hat{k}_5 - \hat{k}_1), \qquad w_3 = k_3$$

$$w_2 = \hat{k}_5 + k_5 * (\hat{k}_5 - \hat{k}_1), \qquad w_1 + w_2 = -k_5.$$

The system of identification for the frequency update is

$$w_2/[(N_1 + 1) * \tau] = \hat{k}_2 + k_2 * (\hat{k}_5 - \hat{k}_1)$$

$$w_3/[(N_1 + 1) * \tau] = k_2$$

$$-(w_1 + w_3)/[(N_2 + 1) * \tau] = \hat{k}_4 + k_4 * (\hat{k}_5 - \hat{k}_1)$$

$$w_3/[(N_2 + 1) * \tau] = k_4$$

$$w_2/[(N_3 + 1) * \tau] = \hat{k}_6 + k_6 * (\hat{k}_5 - \hat{k}_1)$$

$$-(w_1 + w_2)/[(N_3 + 1) * \tau] = k_6.$$

We see that the Kalman filter, which is optimal in the least squares sense, is functionally identical with the NBS time scale algorithm given certain identifications of the weights. There remains the question whether these identifications hold true in practice. Now, the NBS algorithm is adaptive; i.e., the weights vary as the measurements change. We can hope that in steady-state performance of the clocks, i.e., the adaptive weights become constant, the above identifications hold true asymptotically.

For a two-clock realization of the NBS algorithm this is not possible. With only two clocks in the adaptive algorithm, the clock with the least random walk FM has weight 1 in the asymptotic limit. This makes sense theoretically, since with only two clocks there is no way to separate them given only one measurement of their difference. This can be seen mathematically in the equations for the algorithm by assuming a steady-state condition and attempting to solve for the weights. Either they are the same, or one is zero. The asymptotic value for the Kalman gains, from theoretical computations with a frequency Kalman filter and from simulation, are the following functions of the white FM level, σ_ϵ, and random walk FM level, σ_η:

$$K' = [\sigma_{\eta 2}, f * \sigma_{\eta 2}, -\sigma_{\eta 1}, -f * \sigma_{\eta 1}]'/(\sigma_{\eta 1} + \sigma_{\eta 2})$$

where

$$f = \alpha * \big((1 + 4/\alpha)^{1/2} - 1\big)/2$$

and

$$\alpha = (\sigma_{\eta 1}^2 + \sigma_{\eta 2}^2)/(\sigma_{\epsilon 1}^2 + \sigma_{\epsilon 2}^2).$$

A special case of the above results has been published before by Barnes and Stein [5]. If we do fix the weights in the NBS algorithm for equivalence with the Kalman filter equations in the two-clock realization, there are two distributing observations. One is that the weights for time updates in the NBS algorithm become functions of only the random walk FM parameters. This is contrary to intuition and experience with the adaptive NBS algorithm, where the weights are typically proportional to $1/\sigma_\epsilon^2$. Secondly, the exponential filter time constants for frequency update become equal. Again it has been shown that the optimal value for this parameter is a function of σ_ϵ and σ_n for each clock.

For the three-clock realization, the asymptotic value for the Kalman gain is not known in closed form. But from simulation we see that the above problems still apply. For the identification, the weights for time updates would need to be functions of the σ_n values only. However, the values from the Kalman gains appear to be different from those of the adaptive NBS algorithm. Since both are optimal estimators in some sense, we must conclude that they are optimizing somewhat different things.

The difference between the Kalman algorithm and the NBS time scale can be attributed to the nature of what a time scale is. We do not measure time. We measure time differences. We have no measurement of the time error of any clock, only time differences. A time scale does not estimate time; it generates time. A time scale takes clock difference measurements and decides the offset from each of the clocks to define the time of the scale. This decision must be guided by optimality criterion. From the measured time differences we can optimize stability of the scale, since there is enough information in the measurement of three clock differences to estimate clock variances. We also have estimates of the process noise parameters of the clocks from an estimate external to the algorithm. These can be used as guides to limit the deviation of the scale in time. If we wanted only to maintain as accurate a time estimate as possible, intuitively we should simply take the time of the clock with the smallest random walk FM. In fact, this is exactly what the Kalman formalism does. In deriving the Kalman formalism one minimizes the square of the time estimation error to determine the form of the equations. We argue that for a time scale this is not what we want.

III. Frequency Steps in Commercial Cesium Clocks

As mentioned above, it has been shown that a two-parameter stochastic model for the commercial cesium beam clocks in the NBS ensemble is efficient; namely, specifying the white noise FM level and the random walk FM level. The white noise FM is theoretically well understood as to its source and level. The random walk FM is not. Whether this latter FM process has a Gaussian distribution and/or whether the frequency changes occur as discrete steps or gradual changes is not known. Some recent work has indicated some possible causes for frequency steps [7].

There have been indications from some of the clock data that steps in frequency were a possible cause of the above random walk FM being a reasonable model. We have been investigating various efficient methods of frequency step detection. Over the years of running the NBS time scale AT1, from which UTC(NBS), the official time and frequency signal from the Bureau, is derived, we have compensated for these steps in various ways. Big steps are easy to sense. It is the small steps, less than a part in 10^{13}, that can have adverse long-term effects on a time scale, effects that are difficult to detect. The smaller the step, the longer the integration time to detect it, which is contrary to running a real-time scale.

Recently, we have had some indication that because we have monitored these steps to minimize their adverse effect, this effort has netted better long-term frequency stability in our time scales, NBS(AT1) and UTC(NBS) [8]. Without monitoring these steps the long-term stability should be characterized by a random walk FM process, but at a level better than the best contributing clock. Experimental evidence had indicated that monitoring them causes the long-term stability to improve to more like a flicker noise FM process.

Since for integration times from a month to about a year this stability improvement represents a cost savings of the order of a million dollars, we felt it wise to study and document this stability improvement concept more carefully. In other words, if no effort were made to compensate for the random walk FM in the NBS time scale clocks, we would need about four to 16 times as many clocks in order to achieve the same level of stability as with the frequency-step compensation.

To study this concept theoretically was somewhat intractable, so we chose to study it by using simulated data. In this simulation we generate data using a random generator to effect a white FM level of about 3.5 ns at one day in ten clocks with frequency steps occurring at random times of random sizes. The mean time interval between steps was 175 days, with a standard deviation of 40 days. The frequency steps were generated with a mean of 0 and a standard deviation of 1.2 ns/2 h. In attempting to simulate what is actually possible, a clock with a step was allowed to pull the scale off for a period of time until it was "discovered." The time interval until discovery was proportional to the white FM level, and inversely proportional to the actual step size. The constant of proportionality was set to three different values. On one run of the program the steps were discovered after the frequency step produced a time deviation of twice the white FM level. On a second run the steps were discovered after the step pushed the time deviation to three times the white FM level. On a third run, the steps were never discovered. In both cases where steps were discovered, when a step was discovered the sigma for that clock was set large enough to keep it from contributing to the scale. After three times the time constant of the exponential filter for frequency, the sigma was reset to twice what it had been before the step. During the time of deweighting, the adaptive algorithm learned the new frequency of the clock automatically.

Fig. 1 shows the $\sigma_y(\tau)$ plot of four representative simulated clocks in the ensemble. Fig. 2 shows the $\sigma_y(\tau)$ plot of the scale itself, the result of the run where clocks were deweighted after time deviated to $2 * \sigma_\epsilon$. Fig. 3 shows the analogous result, now deweighting clocks after time deviates to $3 * \sigma_\epsilon$. Fig. 4 shows the behavior of the scale when the clocks have not been removed at all after frequency steps. We see from comparing Figs. 2, 3, and 4 that the effect of monitoring the scale and tending to frequency steps can improve the scale at an integration time of 1/3 years (10^7 s) by a factor of 2 to 4.

Since we have been talking about the NBS time scale algorithm acting on a simulated ensemble of clocks, we thought we would conclude with the actual performance of the atomic time scale at NBS, AT1. Fig. 5 gives an estimate of the stability of AT1 obtained from a three-corner hat estimate comparing it with PTB Cs. 1 and an ensemble made up of satellite clocks in the Global Positioning System. The integration times range from ten to 320 days.

IV. Conclusions

The NBS time scale algorithm is an adaptive filter producing a time scale whose short- and long-term stabilities are better than any of its contributing clocks. We have seen that the equations are optimal in the least squares

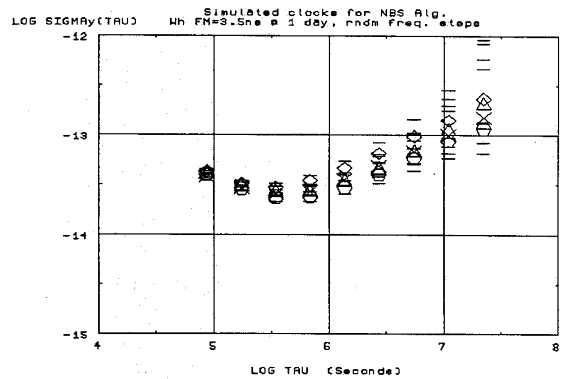

Fig. 1. Performance of four representative simulated clocks. All have a white FM level of 3.5 ns at one day and random frequency steps.

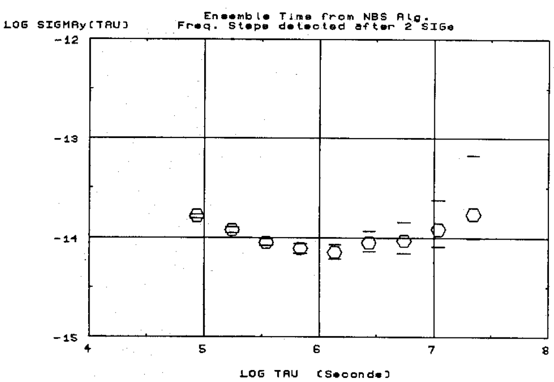

Fig. 2. Performance of the scale, the output of the algorithm, after operating on an ensemble of simulated clocks. Clocks have been removed from the ensemble following a frequency step when the time deviation due to the step corresponds to twice the white FM level.

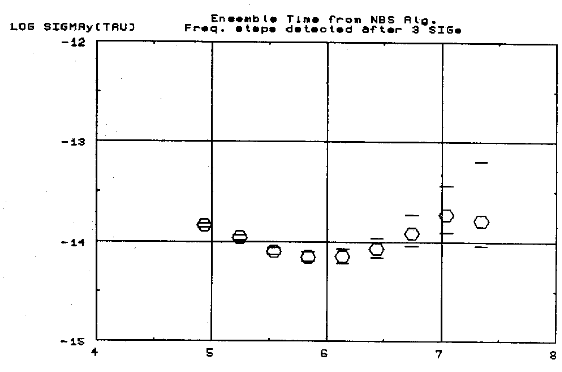

Fig. 3. Performance of the scale, the output of the algorithm, after operating on an ensemble of simulated clocks. Clocks have been removed from the ensemble following a frequency step when the time deviation due to the step corresponds to three times the white FM level.

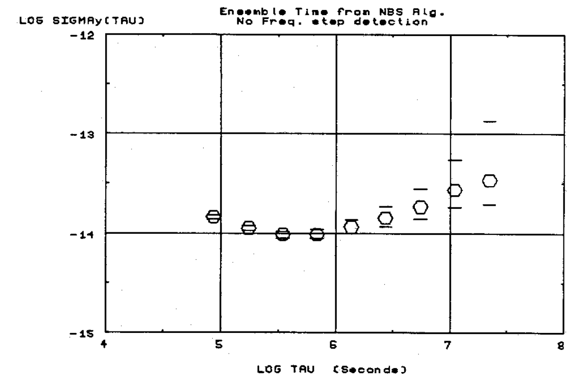

Fig. 4. Performance of the time scale after the algorithm has operated on the simulated clock ensemble with no response to frequency steps.

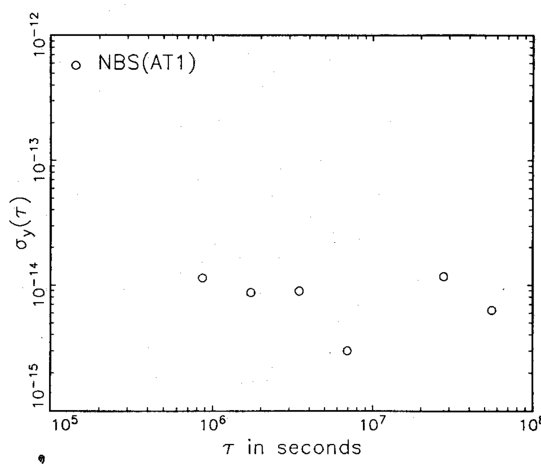

Fig. 5. An estimate of the stability of the actual NBS time scale, AT1, obtained from a three-corner hat estimate comparing it with clock Cs. 1 of the Physikalisch Technische Bundesanstalt, Federal Republic of Germany, and an ensemble made up of satellite clocks in the Global Positioning System. The integration times range from 10 to 320 days.

sense by comparison with the Kalman filter formalism if weights for time update and time constants for frequency updates are identified properly with elements of the Kalman gain vectors. We have noted that the actual weights and time constants used in the NBS algorithm, which have been shown to be optimal, differ from those used in the Kalman filter. The significant difference here is that the Kalman formalism is designed to minimize time error, while the NBS algorithm optimizes time uniformity and frequency stability. Minimization of time error is a meaningless concept.

We have also seen that the way in which an algorithm is maintained can be important. Carefully monitoring the behavior of clocks in an ensemble for frequency steps relative to the scale can improve the stability of the scale at periods of $1/3$ year by a factor of about 2–4. In other words, if no effort were made to compensate for the random walk FM due to observable frequency steps in the NBS time scale clocks, we would need about four to 16 times as many clocks in order to achieve the same level of stability as with the frequency-step compensation.

Appendix

Equations of the NBS Time Scale Algorithm

Definitions

$X_i(t)$, $Y_i(t)$ estimates of time and frequency offsets, respectively, of clock i at time t with respect to some reference time scale;

$\hat{X}_i(t)$	predicted time offset of clock i at time t;
$\hat{Y}_i(t_\tau)$	estimate of frequency of clock i at t over the interval t to $t+\tau$;
$X_{ij}(t)$	measured time difference between clocks i and j at time t;
$\epsilon_i(\tau)$	accumulated error in time estimate of clock i over the interval τ;
$\langle \epsilon_x^2(\tau) \rangle$	mean squared error in ensemble time over the interval τ at time t;
$\langle \ \rangle$	indicates time average;
τ	time interval between measurements;
N_τ	time constant of exponential filter to estimate the current mean squared error;
n	number of clocks in the ensemble;
τ_{mini}	value of τ at minimum $\sigma_y(\tau)$ on Allan variance curve for clock i.

Equations

Time Estimate:

$$\hat{X}_i(t+\tau) = X_i(t) + Y_i(t)\tau. \quad (1)$$

Equation (1) forms a prediction of the time offset for each clock for the next measurement time $(t+\tau)$ based on the current estimates of time and filtered frequency.

$$X_j(t+\tau) = \sum_{i=1}^{n} w_i(\tau)\left[\hat{X}_i(t+\tau) - X_{ij}(t+\tau)\right]. \quad (2)$$

Equation (2) estimates the time offset of each clock j at time $t+\tau$ given the measurements $X_{ij}(t+\tau)$.

Frequency Estimate:

$$\hat{Y}_i(t+\tau) = \frac{X_i(t+\tau) - X_i(t)}{\tau}. \quad (3)$$

Equation (3) estimates the average frequency of each clock over the interval τ based on the latest two estimates of X_i.

$$Y_i(t+\tau) = \frac{1}{m_i+1}\left[\hat{Y}_i(t+\tau) + m_i Y_i(t)\right]. \quad (4)$$

Equation (4) incorporates past measurements into an exponentially filtered estimate of the current average frequency of clock i. The exponential frequency-weighting time constant (m_i) is determined from the relative levels of white noise and random walk (or flicker) FM for clock i (5).

$$m_i = \frac{1}{2}\left(-1 + \left(\frac{1}{3} + \frac{4\tau_{\text{mini}}^2}{3\tau^2}\right)^{1/2}\right). \quad (5)$$

Equation (5) computes m_i used in equation 4 to form the filtered estimate of the frequency of clock i. This value of m_i can be shown to minimize the error in predicting time (1) given two kinds of noise in the clock (white and random walk FM). If white FM and flicker FM are more suitable models, then m_i can be approximated as τ_x/τ, where τ_x is the intercept value of τ on a $\sigma_y(\tau)$ plot for the white and flicker FM.

Weight Estimates:

$$\epsilon_i(\tau) = \left|\hat{X}_i(t+\tau) - X_i(t+\tau)\right| + K_i. \quad (6)$$

Equation (6) is the accumulated error in the estimate of X_i over the interval τ. The additive term K_i accounts for the fact that the term in brackets on the right-hand side of (6) is biased because clock i is part of the ensemble. See (10) to calculate K_i.

$$\langle \epsilon_i^2(\tau) \rangle_{t+\tau} = \frac{1}{N_\tau + 1}\left[\epsilon_i^2(\tau) + N_\tau \langle \epsilon_i^2(\tau) \rangle_t\right]. \quad (7)$$

Equation (7) is an exponential time filter for the determination of the mean squared time error of each clock. Since the noise characteristics of a cesium clock may not be stationary, past measurements are deweighted in the averaging process. The time constant for the filter is typically chosen to be $N_\tau = 20$ days. The initial value of $\langle \epsilon_i^2(\tau) \rangle$ can be estimated as $\tau^2\sigma_y^2(\tau)$.

$$\langle \epsilon_x^2(\tau) \rangle = \left(\sum_{i=1}^{n} \frac{1}{\epsilon_i^1(\tau)}\right)^{-1}. \quad (8)$$

Equation (8) forms an estimate of ensemble time error. Any clock can only improve this number—a poorly performing clock cannot harm the stability of the ensemble.

$$w_i = \frac{\langle \epsilon_x^2(\tau) \rangle}{\langle \epsilon_i^2(\tau) \rangle}. \quad (9)$$

Equation (9) calculates the weight to be used in (2) for each clock. When calculated this way, the resulting error in ensemble time with respect to a perfect clock can be shown to be minimized in a least squares sense.

$$K_i = \frac{.8\langle \epsilon_x^2 \rangle}{\langle \epsilon_i^2 \rangle^{1/2}}. \quad (10)$$

Equation (10) estimates the bias in the error estimates from the first term on the right of (6). This error estimate is biased small, on the average, because each clock is a member of the ensemble and sees itself through its weighting factors. The larger a clock's weight, the larger the bias. Under the assumption of a normal distribution of errors the size of the bias can be estimated as given by (10), which is added to (5) in order to remove the bias, on the average [6].

References

[1] D. W. Allan and M. A. Weiss, *The NBS Atomic Time Scale Algorithm: AT1*, NBS Tech. Note 1316, 1989.

[2] R. H. Jones and P. V. Tryon, "Continuous time series models for unequally spaced data applied to modeling atomic clocks," *SIAM J. Sci. Stat. Comput.*, vol. 8, no. 1, pp. 71–81, Jan. 1987.

[3] J. Barnes, "Time scale algorithms using Kalman filters—Insights from simulation," presented at *2nd Symp. Atomic Time Scale Algorithms*, National Bureau of Standards, Boulder, CO, June 23–25, 1982.

[4] S. Stein, "Ensemble averaging and the control of precision oscillators and clocks," Rep. under contract no. N00173-87-M-J780, for Naval Research Laboratory, Code 8320, Washington, DC, 1988.

[5] J. A. Barnes and S. R. Stein, "Application of Kalman filters and ARIMA models to digital frequency and phase lock loops," in *19th Proc. PTTI*, Dec. 1987, pp. 311–323.

[6] M. A. Weiss and D. W. Allan, "An NBS calibration procedure for providing time and frequency at a remote site by weighting and smoothing of GPS common view data," *IEEE Trans. Instrum. Meas.*, vol. IM-36. pp. 572–579, June 1987.

[7] J. E. Gray, H. E. Machlan, and D. W. Allan, "The effect of humidity on commercial cesium beam atomic clocks," presented at 42nd Annual Frequency Control Symp., 1988.

[8] D. W. Allan, "A study of long-term stability of atomic clocks," in *Proc. 19th Annual PTTI Planning Meeting*, 1987.